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On the integral closures of certain ideals generated by regular sequences

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1 Introduction

The purpose of this report is to introduce a notion of equimultiplicity for filtrations in local rings. We will apply it's theory for computation of the integral closures of certain ideals generated by regular sequences.

Throughout this report A is a d-dimensional local ring with the maximal ideal $\mathfrak{m}$ and a family of ideals $\mathcal{F}=\{F_n\}_{n\in\mathbb{Z}}$ is a filtration in A, which means (i) $F_n\supseteq F_{n+1}$ for all $n\in\mathbb{Z}$, (ii) $F_0=A$, $F_1\neq A$ and (iii) $F_mF_n\subseteq F_{m+n}$ for all $m,n\in\mathbb{Z}$. We can define the following graded algebras associated to a filtration $\mathcal{F}$.

$$\begin{aligned}\mathrm{R}(\mathcal{F}) &= \sum_{n\geq 0}F_nt^n\subseteq A[t]\,,\\ \mathrm{R}'(\mathcal{F}) &= \sum_{n\in\mathbb{Z}}F_nt^n\subseteq A[t,t^{-1}]\ \text{and}\\ \mathrm{G}(\mathcal{F}) &= \mathrm{R}'(\mathcal{F})/t^{-1}\mathrm{R}'(\mathcal{F})=\oplus_{n\geq 0}F_n/F_{n+1},\end{aligned}$$

where t is an indeterminate. These algebras are respectively called the Rees algebra of $\mathcal{F}$, the extended Rees algebra of $\mathcal{F}$ and the associated graded ring of $\mathcal{F}$. We always assume that $\mathrm{R}(\mathcal{F})$ is Noetherian and $\dim\mathrm{R}(\mathcal{F})=d+1$.

2 The analytic spread of a filtration

We set $\ell(\mathcal{F})=\dim A/\mathfrak{m}\otimes_A\mathrm{R}(\mathcal{F})$ and call it the analytic spread of $\mathcal{F}$. It is easy to see that $\ell(\mathcal{F})=\dim A/\mathfrak{m}\otimes_A\mathrm{G}(\mathcal{F})$. We say that a system of elements $a_1,\cdots,a_r$ in A is a reduction of $\mathcal{F}$, if the following condition $(*)$ is satisfied.

$(*)$ There exist $m_i>0$ for all $1\leq i\leq r$ such that $a_i\in F_{m_i}$ and $F_n=\sum_{i=1}^{r}a_iF_{n-m_i}$ for all $n\gg 0$.

This condition is equivalent to saying that we have a module-finite extension

$$A[a_1t^{m_1}, \cdots, a_rt^{m_r}] \subseteq \mathrm{R}(\mathcal{F})$$

of rings. If $a_1, \cdots, a_r$ is a reduction of $\mathcal{F}$, then obviously we have $\ell(\mathcal{F}) \leq r$. We say that a reduction $a_1, \cdots, a_r$ of $\mathcal{F}$ is minimal, if $\ell(\mathcal{F}) = r$. We always have a minimal reduction for any filtration $\mathcal{F}$ (It is not necessary to assume that the residue field is infinite).

By the definition of filtration, we have $\sqrt{F_n} = \sqrt{F_1}$ for all $n \geq 1$, and so $\mathrm{ht}_A F_n$ is constant for $n \geq 1$. We denote this number by $\mathrm{ht}_A \mathcal{F}$. Then the follwing inequality always holds:

$$\mathrm{ht}_A \mathcal{F} \leq \ell(\mathcal{F}) \leq \dim A.$$

We say that $\mathcal{F}$ is eqyimultiple, if $\mathrm{ht}_A \mathcal{F} = \ell(\mathcal{F})$. If $\mathcal{F}$ is equimultiple and $a_1, \cdots, a_r$ is a minimal reduction of $\mathcal{F}$, the number m_i in $(*)$ must coincide to

$$\deg_{\mathcal{F}} a_i := \max\{n \mid a_i \in F_n\}$$

for all $1 \leq i \leq r$.

Example 2.1 *Let $\mathfrak{p}$ be a prime ideal in A such that $\dim A/\mathfrak{p} = 1$. Let $F_n = \mathfrak{p}^{(n)}$ for all $n \in \mathbb{Z}$, where $\mathfrak{p}^{(n)}$ denotes the n-th symbolic power of $\mathfrak{p}$. If $\mathrm{R}(\mathcal{F})$ is Noetherian, then $\mathcal{F}$ is equimultiple.*

Proof. Because $\mathrm{R}(\mathcal{F})$ is Noetherian, there exists a positive integer k such that $\mathfrak{p}^{(kn)} = [\mathfrak{p}^{(k)}]^n$ for all $n \in \mathbb{Z}$. This means the k-th Veronesean subring $\mathrm{R}(\mathcal{F})^{(k)} = \sum_{n\geq 0} \mathfrak{p}^{(kn)}t^{kn}$ is isomorphic to $\mathrm{R}(\mathfrak{p}^{(k)})$ and $\operatorname{depth} A/[\mathfrak{p}^{(k)}]^n = 1$ for all $n \geq 1$. Then the extension

$$\mathrm{R}(\mathfrak{p}^{(k)}) \subseteq \mathrm{R}(\mathcal{F})$$

is module-finite and $\ell(\mathfrak{p}^{(k)}) = d - 1$ by Burch's inequality. Let $a_1, \cdots, a_{d-1}$ be a minimal reduction of $\mathfrak{p}^{(k)}$. Then the extension

$$A[a_1t^k, \cdots, a_{d-1}t^k] \subseteq \mathrm{R}(\mathcal{F})^{(k)}$$

is module-fininite, and so

$$A[a_1, \cdots, a_{d-1}] \subseteq \mathrm{R}(\mathcal{F})$$

is also module-finite.

Example 2.2 *Let J be an ideal in A generated by a subsystem of parameters $a_1, \cdots, a_s$ for A. Let $\mathcal{F}$ be a filtration such that $J^n \subseteq F_n \subseteq \overline{J^n}$ for all $n \in \mathbb{Z}$. If $\mathrm{R}(\mathcal{F})$ is Noetherian, then $\mathcal{F}$ is equimultiple and $a_1, \cdots, a_s$ is a minimal reduction of $\mathcal{F}$.*

Proof. Obviously, $\mathrm{ht}_A \mathcal{F} = s$. As $J^n \subseteq F_n$ for all $n \in \mathbb{Z}$, $\mathrm{R}(\mathcal{F})$ contains $A[a_1 t, \cdots, a_s t]$. Moreover, as $F_n \subseteq \overline{J^n}$ for all $n \in \mathbb{Z}$, $\mathrm{R}(\mathcal{F})$ is integral over $A[a_1 t, \cdots, a_s t]$. Because $\mathrm{R}(\mathcal{F})$ is Noetherian, we see that the extension

$$A[a_1 t, \cdots, a_s t] \subseteq \mathrm{R}(\mathcal{F})$$

is module-finite.

For a prime ideal $\mathfrak{p}$ in A containing F_1, we set $\mathcal{F}_\mathfrak{p} = \{F_n A_\mathfrak{p}\}_{n \in \mathbb{Z}}$, which is a filtration in $A_\mathfrak{p}$. Obviously, $\ell(\mathcal{F}_\mathfrak{p}) \leq \ell(\mathcal{F})$ for any prime ideal $\mathfrak{p}$ in A containing F_1.

3 Cohen-Macaulay property of the graded rings associated to equimultiple filtrations

Theorem 3.1 *Let A be a quasi-unmixed local ring. If $\mathcal{F}$ is equimultiple, then we have*

$$\mathrm{a}(\mathrm{G}(\mathcal{F})) = \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Assh}_A A/F_1\}$$

Theorem 3.2 *Let A be a Cohen-Macaulay ring. Let $\mathcal{F}$ be an equimultiple filtration. We set $s = \mathrm{ht}_A \mathcal{F}$. Then the following conditions are equivalent:*

(1) $\mathrm{G}(\mathcal{F})$ *is a Cohen-Macaulay ring.*

(2) $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ *is Cohen-Macaulay for all* $\mathfrak{p} \in \mathrm{Assh}_A A/F_1$ *and there exists a minimal reduction* $a_1, \cdots, a_s$ *of* $\mathcal{F}$ *such that* $A/(a_1, \cdots, a_s) + F_n$ *is Cohen-Macaulay for all* $1 \leq n \leq \mathrm{a}(\mathrm{G}(\mathcal{F})) + \sum_{i=1}^{s} \deg_\mathcal{F} a_i$.

When this is the case, for any minimal reduction $b_1, \cdots, b_s$ of $\mathcal{F}$, $A/(b_1, \cdots, b_s) + F_n$ is Cohen-Macaulay for all $n \geq 1$ and

$$\mathrm{R}(\mathcal{F}) = A[\{b_i t^{\deg_\mathcal{F} b_i}\}_{1 \leq i \leq s}, \{F_n t^n\}_{1 \leq n \leq \mathrm{a}(\mathrm{G}(\mathcal{F})) + \sum_{i=1}^{s} \deg_\mathcal{F} b_i}].$$

Corollary 3.3 *Let A be a Cohen-Macaulay ring. Let I be an equimultiple ideal. Then the following conditions are nequivalent:*

(1) $\mathrm{G}(I)$ *is a Cohen-Macaulay ring.*

(2) $\mathrm{G}(I_\mathfrak{p})$ *is Cohen-Macaulay for all* $\mathfrak{p} \in \mathrm{Assh}_A A/I$ *and there exists a minimal reduction* J *of* I *such that* $A/J + I^n$ *is Cohen-Macaulay for all* $1 \leq n \leq \mathrm{r}_J(I)$.

4 Integral closures of certain ideals

Applying the results in section 3, we can prove the following assertions.

Example 4.1 *Let $A = k[[X, Y, Z]]$ be the formal power series ring over a field k. Suppose that the ideal generated by the maximal minors of the matrix*

$$\begin{pmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{pmatrix}$$

is a prime ideal, where $\alpha, \beta, \gamma, \alpha', \beta'$ and γ' are all positive integers. We put $a = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}, b = X^{\alpha+\alpha'} - Y^{\beta}Z^{\gamma'}$ and $c = Y^{\beta+\beta'} - X^{\alpha}Z^{\gamma}$. Let $J = (a, b)A$. Then we have

$$\overline{J^n} = J^{n-1} \cdot (a, b, \{X^iZ^jc \mid i, j \geq 0 \text{ and } i/\alpha' + j/\gamma' \geq 1\})A$$

for all $n \geq 1$ and $\overline{\mathrm{R}(J)}$ is a Cohen-Macaulay ring.

Example 4.2 *Let $A = k[[X, Y, Z, W]]$ be the formal power series ring over a field k. Let α, β and γ be positive integers such that $0 < \alpha \leq \beta \leq \gamma$. We set*

$$a = X^{\alpha+\ell} - Y^{\beta}W, b = Y^{\beta+m} - Z^{\gamma}W, c = Z^{\gamma+1} - X^{\alpha}W \text{ and } d = W^3 - X^{\ell}Y^mZ,$$

where $\ell = \gamma + \beta - 2\alpha + 1$ and $m = 2\gamma - \beta - \alpha + 1$. It is easy to see that a, b, c is a regular sequence in A. Let $J = (a, b, c)A$. Then we have

$$\begin{aligned} \overline{J} &= J + (\{X^iY^jZ^kd \mid i/\alpha + j/\beta + k/\gamma \geq 1\})A\,, \\ \overline{J^2} &= \overline{J}^2 + (X^iY^jZ^kd^2 \mid i/2\alpha + j/2\beta + k/2\gamma \geq 1\})A \text{ and} \\ \overline{J^n} &= \overline{J}^{n-2} \cdot \overline{J^2} \text{ for } n \geq 3. \end{aligned}$$

Moreover $\overline{\mathrm{R}(J)}$ is a Cohen-Macaulay ring.

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